

# COMPOSITIONS OF $n$ SATISFYING SOME COPRIMALITY CONDITIONS

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**ABSTRACT.** An  $\ell$ -composition of  $n$  is a sequence of length  $\ell$  of positive integers summing up to  $n$ . In this paper, we investigate the number of  $\ell$ -compositions of  $n$  satisfying two natural coprimality conditions. Namely, we first give an exact asymptotic formula for the number of  $\ell$ -compositions having the first summand coprime to the others. Then, we estimate the number of  $\ell$ -compositions whose summands are all pairwise coprime.

## 1. INTRODUCTION

Given a positive integer  $n \in \mathbb{N}$ , in this paper we are interested on the size of two sets of *compositions* of  $n$  both satisfying some natural coprimality conditions. For  $k \geq 1$ , the first set consists of the  $(k+1)$ -compositions  $(x, y_1, \dots, y_k)$  of  $n$  with  $x$  coprime to  $y_i$ , for each  $i \in \{1, \dots, k\}$ . We denote this set by  $\mathcal{A}_k(n)$  and its size by  $A_k(n)$ , that is,

$$\mathcal{A}_k(n) = \{(x, y_1, \dots, y_k) \in \mathbb{N}^{k+1} : n = x + y_1 + \dots + y_k, \gcd(x, y_1 \cdots y_k) = 1\},$$

$$A_k(n) = \#\mathcal{A}_k(n).$$

Observe that  $\mathcal{A}_k(n) = \emptyset$  when  $n < k+1$  and that  $\mathcal{A}_k(n)$  is a singleton if  $n = k+1$ . Thus, we will assume that  $n > k+1$ . In particular if  $k \geq 2$  we will assume that  $n \geq 4$ .

For  $k \geq 2$ , the second set consists of the  $k$ -compositions  $(x_1, \dots, x_k)$  of  $n$  with  $x_i$  coprime to  $x_j$ , for every two distinct elements  $i, j \in \{1, \dots, k\}$ . We denote this set by  $\mathcal{B}_k(n)$  and its size by  $B_k(n)$ , that is,

$$\mathcal{B}_k(n) = \{(x_1, \dots, x_k) \in \mathbb{N}^k : n = x_1 + \dots + x_k, \gcd(x_i, x_j) = 1, 1 \leq i < j \leq k\},$$

$$B_k(n) = \#\mathcal{B}_k(n).$$

Since for  $n = k$ , the set  $\mathcal{B}_k(n)$  is a singleton, dealing with  $\mathcal{B}_k(n)$  we will assume that  $n > k$ . Our goal is to give an exact asymptotic estimate for  $A_k(n)$  and  $B_k(n)$ , as functions of  $n$  and  $k$ . We clearly have  $A_1(n) = B_2(n) = \varphi(n)$  (the Euler totient function) and the asymptotic behaviour of  $\varphi(n)$  is well-understood. Before stating our main results we need the following definition. Throughout the paper, we use  $p$  and  $q$  for primes.

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2000 *Mathematics Subject Classification.* 05A17, 11P81.

*Key words and phrases.* compositions; coprime summand; pairwise coprime.

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The first author is supported by the MIUR project “Teoria dei gruppi ed applicazioni”. The third author is partially supported by the University of Western Australia as part of the Federation Fellowship project FF0776186.

**Definition 1.** For positive integers  $k$  and  $n$ , define

$$\begin{aligned}\psi_k(x) &= \frac{x^k - (x-1)^k + (-1)^k}{x}, & \delta_k(x) &= \frac{(x-1)^k + k(x-1)^{k-1} + (-1)^k(k-1)}{x}, \\ C_k &= \prod_p \left(1 - \frac{\psi_k(p)}{p^k}\right), & D_k &= \prod_p \frac{\delta_k(p)}{p^{k-1}}, \\ f_k(n) &= \prod_{p|n} \left(1 + \frac{(-1)^k}{p^k - \psi_k(p)}\right), & g_k(n) &= \prod_{p|n} \left(1 + \frac{(-1)^{k-1}(k-1)}{\delta_k(p)}\right).\end{aligned}$$

Our first main result is the following.

**Theorem 2.** For  $k \geq 1$ , we have the estimate

$$\left| A_k(n) - C_k f_k(n) \frac{n^k}{k!} \right| \leq \frac{2+e}{\sqrt{2\pi k}} (e^2 \log n)^k n^{k-1}.$$

In Theorem 2 and in what follows, we could use the Landau symbol  $O$  with its usual meaning. However, we usually shall avoid the symbol  $O$  because we want our estimates to be completely explicit. Throughout the proofs we shall use  $\theta$  (with or without subscripts) for a real number with  $|\theta| \leq 1$ .

In view of Theorem 2 we have that the leading term of  $A_k(n)$  is  $n^k/k!$  multiplied by  $C_k$  (which depends only on  $k$ ) and by  $f_k(n)$  (which depends upon the prime factorization of  $n$ ).

When  $k = 1$ , since  $\psi_1(x) = 0$ , we have that  $C_1 = 1$  and

$$C_1 f_1(n) n = \prod_{p|n} \left(1 - \frac{1}{p}\right) n = \varphi(n).$$

So, the leading term in Theorem 2 actually equals  $A_1(n)$ .

Our next result collects some information on  $C_k$  and on  $f_k(n)$  which, together with Theorem 2, unravels the asymptotic behaviour of  $A_k(n)$ .

**Theorem 3.** For every  $k \geq 2$ , the series  $C_k$  converges and  $0 < C_k < 1$ . Furthermore  $2/3 < f_k(n) < 2$ .

For  $B_k(n)$ , we prove the following.

**Theorem 4.** For  $k \geq 2$  and  $n \geq e^{k2^{k+2}}$ , we have the estimate

$$\left| B_k(n) - D_k g_k(n) \frac{n^{k-1}}{(k-1)!} \right| \leq \frac{707 n^{k-1}}{\log n}.$$

Exactly as in Theorem 2, we see that the leading term of  $B_k(n)$  is  $n^{k-1}/(k-1)!$  multiplied by  $D_k$  (which depends only on  $k$ ) and by  $g_k(n)$  (which depends upon the prime factorization of  $n$ ).

When  $k = 2$ , since  $\delta_2(x) = x$ , we have that  $D_2 = 1$  and

$$D_2 g_2(n) n = \prod_{p|n} \left(1 - \frac{1}{p}\right) n = \varphi(n).$$

So the leading term in Theorem 4 actually equals  $B_2(n)$ .

Theorem 5 collects some information on  $D_k$  and  $g_k(n)$ , which helps to describe the order of magnitude of  $B_k(n)$ .

**Theorem 5.** *For every  $k \geq 3$ , the series  $D_k$  converges and  $0 < D_k < 1$ . Furthermore  $1/2k < g_k(n) < 2k$ .*

Similar problems on compositions with restricted arithmetical conditions have been studied in [8] and [11]. In particular, using the principle of inclusion-exclusion, Gould [8, Theorem 5] has obtained a formula for the number of  $k$ -compositions  $(x_1, \dots, x_k)$  of  $n$  with  $\gcd(x_1, x_2, \dots, x_k) = 1$ .

Finally, in Table 1, we give some approximate values for  $C_k$  and  $D_k$  (for  $1 \leq k \leq 7$ ), which are obtained with the help of `magma` [4].

$k$	$C_k$	$D_k$
1	1	—
2	0.32263	1
3	0.38159	0.12548
4	0.26778	0.19680
5	0.26328	0.01312
6	0.23051	0.02330
7	0.22123	0.00099

TABLE 1. Some values for  $C_k$  and  $D_k$

**1.1. Applications to Group Theory and to Galois Theory.** In [5], the first author together with Praeger, investigated the *normal coverings* of a finite group  $G$ , that is, the families  $H_1, \dots, H_r$  of proper subgroups of  $G$  such that each element of  $G$  has a conjugate in  $H_i$ , for some  $i \in \{1, \dots, r\}$ . The minimum  $r$  is usually denoted by  $\gamma(G)$ . They find that when  $G$  is the symmetric group  $S_n$  or the alternating group  $A_n$ , the number  $\gamma(G)$  lies between  $a\varphi(n)$  and  $bn$  for certain positive constants  $a$  and  $b$ . More recently, Bubboloni, Spiga and Praeger [6] have developed some new research on this topic starting with the idea that primitive subgroups of the symmetric group are “few and small” (see [2], [9], [10] and [12]) and therefore cannot play a significant role in normal coverings. With an application of Theorem 2, they show that, for  $G = S_n$  or  $A_n$ , the number  $\gamma(G)$  is asymptotically linear in  $n$  (improving every previous result in this area).

The normal coverings of the symmetric and of the alternating group are relevant for some problems in Galois theory [5]. Let  $f(x) \in \mathbb{Z}[x]$  be a polynomial which has a root mod  $p$ , for all primes  $p$ , and consider its Galois group over the rationals  $G = \text{Gal}_{\mathbb{Q}}(f)$ . Let  $f_1(x), \dots, f_k(x) \in \mathbb{Z}[x]$  be the distinct irreducible factors of  $f(x)$  over  $\mathbb{Q}$ , and suppose that no  $f_i$  is linear. By [3, Theorem 2], we have  $k \geq \gamma(G)$ . In other words, for a polynomial  $f(x)$  which has a root mod  $p$ , for all primes  $p$ , but no root in  $\mathbb{Q}$ , the number of subgroups involved in a minimal normal covering of its Galois group is a lower bound for the number of distinct irreducible factors of  $f(x)$  over  $\mathbb{Q}$ . In this context the pertinence of the results in [5], in this paper and in [6] relies on the fact that the most common Galois groups are the symmetric and the alternating groups [15].

Finally, we point out that Theorems 2 and 3 are also used in [7], to obtain some bounds on the diameter of the generating graph of  $S^n$ , for  $n \geq 1$  and for a finite non-abelian simple group  $S$ .

**1.2. Structure of the paper.** Theorems 3 and 5 are proved in Section 3, Theorem 2 is proved in Section 4 and Theorem 4 is proved in Section 5.

## 2. EN ROUTE TO THE PROOF OF THEOREM 2

We denote with

$$\mathcal{K}_k(n) = \{(x_1, x_2, \dots, x_{k+1}) \in \mathbb{N}^{k+1} : n = \sum_{i=1}^{k+1} x_i\} \text{ and}$$

$$\mathcal{U}_k(n) = \{(x_1, x_2, \dots, x_{k+1}) \in (\mathbb{N} \cup \{0\})^{k+1} : n = \sum_{i=1}^{k+1} x_i\},$$

respectively, the set of  $(k+1)$ -compositions and the set of *generalized*  $(k+1)$ -compositions of  $n$ . It is well known that

$$(1) \quad \begin{aligned} K_k(n) = \#\mathcal{K}_k(n) &= \binom{n-1}{k} = \frac{n^k}{k!} + \theta_K k n^{k-1}, \text{ and} \\ U_k(n) = \#\mathcal{U}_k(n) &= \binom{n+k}{k} = \frac{n^k}{k!} + \theta_U k n^{k-1} \end{aligned}$$

(see for instance [8]).

The following definition will turn out to be crucial in the proof of Theorem 2.

**Definition 6.** For a square-free positive integer  $d \geq 1$ , write

$$\mathcal{K}_{k,d}(n) = \{(x, y_1, \dots, y_k) \in \mathcal{K}_k(n) : d \text{ divides } \gcd(x, \prod_{i=1}^k y_i)\}$$

and

$$K_{k,d}(n) = \#\mathcal{K}_{k,d}(n).$$

Note that, if  $J = \{p_1, \dots, p_s\}$  is a set of primes and  $d = p_1 \cdots p_s$ , then

$$\mathcal{K}_{k,d}(n) = \bigcap_{i=1}^s \mathcal{K}_{k,p_i}(n).$$

Moreover  $K_{k,1}(n) = K_k(n)$ . Clearly  $\mathcal{K}_{k,d}(n)$  is empty when  $d > n$ .

Our idea to compute  $A_k(n)$  is to use the principle of inclusion-exclusion as

$$\mathcal{A}_k(n) = \mathcal{K}_k(n) \setminus \bigcup_{p \in \mathbb{P}_n} \mathcal{K}_{k,p}(n),$$

where

$$\mathbb{P}_n = \{r \in \mathbb{N} : r \text{ prime, } r \leq n\}.$$

Namely,

$$\begin{aligned} A_k(n) &= \#\mathcal{K}_k(n) - \# \left( \bigcup_{p \in \mathbb{P}_n} \mathcal{K}_{k,p}(n) \right) = K_k(n) + \sum_{\emptyset \neq J \subseteq \mathbb{P}_n} (-1)^{\#J} \# \bigcap_{p \in J} \mathcal{K}_{k,p}(n) \\ (2) \quad &= \sum_{J \subseteq \mathbb{P}_n} (-1)^{\#J} \# \bigcap_{p \in J} \mathcal{K}_{k,p}(n) = \sum_{1 \leq d \leq n} \mu(d) K_{k,d}(n), \end{aligned}$$

where  $\mu$  is the Möbius function.

In light of (2), to prove Theorem 2 we need to estimate the numbers  $K_{k,d}(n)$ . This will be possible thanks to some lemmas on linear equations modulo  $p$  which we give in Section 3. An asymptotic formula for  $K_{k,d}(n)$  is then obtained in Proposition 11. Throughout the rest of this paper, we reserve the letter  $d$  to denote a square-free positive integer.

3. LINEAR EQUATIONS MODULO  $p$  AND THE PROOF OF THEOREMS 3 AND 5

We start by introducing two auxiliary polynomials  $\phi_k(x)$  and  $\eta_k(x)$  which are closely related to  $\psi_k(x)$  and  $\delta_k(x)$  in Definition 1. These polynomials turn out to be fundamental for understanding the *local* aspects of the sets  $\mathcal{A}_k(n)$  and  $\mathcal{B}_k(n)$ .

**Definition 7.** For  $k \geq 1$ , define

$$\begin{aligned}\phi_k(x) &= \frac{x^k - (x-1)^k + (-1)^{k+1}(x-1)}{x}, \\ \eta_k(x) &= \frac{(x-1)^k + k(x-1)^{k-1} + (-1)^{k-1}(k-1)(x-1)}{x}.\end{aligned}$$

A direct calculation shows immediately that for any  $k \geq 1$ , we have:

$$(3) \quad \begin{aligned}\phi_k(x) &= \psi_k(x) + (-1)^{k+1}, \\ \eta_k(x) &= \delta_k(x) + (-1)^{k-1}(k-1).\end{aligned}$$

When  $k = 1$ , we have

$$\psi_1(x) = 0, \quad \delta_1(x) = 1, \quad \phi_1(x) = 1, \quad \eta_1(x) = 1.$$

**Lemma 8.** Let  $k$  and  $n$  be integers with  $k \geq 1$  and  $n \geq 0$ . Write

$$\begin{aligned}S_k(x) &= x^{k-1} - \phi_k(x) = \frac{(x-1)^k + (-1)^k(x-1)}{x}, \\ W_k(x) &= x^{k-1} - \psi_k(x) = \frac{(x-1)^k + (-1)^{k+1}}{x}.\end{aligned}$$

Then:

a) The number of solutions of

$$(4) \quad y_1^* + \cdots + y_k^* \equiv n \pmod{p},$$

with  $y_i^* \in \{0, \dots, p-1\}$  for every  $i \in \{1, \dots, k\}$  and with  $y_j^* = 0$  for some  $j \in \{1, \dots, k\}$ , is  $\phi_k(p)$  if  $p \mid n$  and  $\psi_k(p)$  if  $p \nmid n$ .

b) For  $k \geq 2$ , the inequality

$$\max\{\phi_k(x), \psi_k(x)\} \leq kx^{k-2}$$

holds for all  $x \geq 2$ .

c) The number of solutions of (4), with  $y_j^* \in \{1, \dots, p-1\}$  for every  $i \in \{1, \dots, k\}$ , is  $S_k(p)$  if  $p \mid n$ , and  $W_k(p)$  if  $p \nmid n$ .

d) The number of solutions of (4), with  $y_j^* \in \{0, \dots, p-1\}$  for every  $i \in \{1, \dots, k\}$  and with  $y_j^* = 0$  for at most one  $j \in \{1, \dots, k\}$ , is  $\eta_k(p)$  if  $p \mid n$ , and  $\delta_k(p)$  if  $p \nmid n$ .

e) For  $k \geq 1$ , we have

$$\max\{\psi_k(p), \delta_k(p), \phi_k(p), \eta_k(p)\} \leq p^{k-1}.$$

Moreover, if  $k \geq 3$ , then

$$(p-1)^{k-1} \leq \delta_k(p) < p^{k-1}.$$

f) For  $k \geq 2$ , we have

$$\min\{\psi_k(p), \delta_k(p), \phi_k(p), \eta_k(p)\} \geq 1.$$

g) For  $k \geq 3$ , the inequality

$$\max \left\{ \frac{p^{k-1}}{\eta_k(p)}, \frac{p^{k-1}}{\delta_k(p)} \right\} \leq 1 + \frac{k2^k}{p^2}$$

holds for all  $p \geq \sqrt{k2^k}$ .

*Proof.* a) We prove a) by induction on  $k$ . If  $k = 1$ , then the result is obvious because  $\phi_1(p) = 1$  and  $\psi_1(p) = 0$ . Assume that  $k \geq 2$ . Suppose that  $p \mid n$ . Now (4) has  $p^{k-2}$  solutions with  $y_1^* = 0$ . Also, for any  $y_1^* \neq 0$ , (4) is equivalent to  $y_2^* + \dots + y_k^* \equiv (-y_1^*) \pmod{p}$  and so, by induction, there exist  $\psi_{k-1}(p)$  solutions of (4) having at least one coordinate being zero and with a fixed  $y_1^* \neq 0$ . Summing up, the number of solutions of (4) with at least one coordinate being zero is

$$\begin{aligned} p^{k-2} + (p-1)\psi_{k-1}(p) &= p^{k-1} + (p-1) \frac{p^{k-1} - (p-1)^{k-1} + (-1)^{k-1}}{p} \\ &= \frac{p^k - (p-1)^k + (-1)^{k+1}(p-1)}{p} = \phi_k(p). \end{aligned}$$

Suppose that  $p \nmid n$ . The number of solutions of (4) with  $y_1^* = 0$  is  $p^{k-2}$ . If  $y_1^* \in \{0, \dots, p-1\}$  and  $y_1^* \equiv n \pmod{p}$ , then (4) is equivalent to  $y_2^* + \dots + y_k^* \equiv 0 \pmod{p}$ , which, by induction, has  $\phi_{k-1}(p)$  solutions with at least one coordinate being zero. Finally, for any  $y_1^* \not\equiv 0, n \pmod{p}$ , (4) is equivalent to  $y_2^* + \dots + y_k^* \equiv (n - y_1^*) \pmod{p}$  which, again by induction, has  $\psi_{k-1}(p)$  solutions with at least one coordinate being zero. Summing up, the number of solutions of (4) with at least one coordinate being zero is

$$p^{k-2} + \phi_{k-1}(p) + (p-2)\psi_{k-1}(p) = \frac{p^k - (p-1)^k + (-1)^k}{p} = \psi_k(p).$$

b) If  $k = 2$ , we have  $\phi_2(x) = 1 < 2$  and  $\psi_2(x) = 2 \leq 2$ . Now assume that  $k \geq 3$ . From the factorization,  $u^k - v^k = (u-v)(u^{k-1} + u^{k-2}v + \dots + uv^{k-2} + v^{k-1})$ , we see that

$$\begin{aligned} \phi_k(x) &= \frac{1}{x} (x^{k-1} + x^{k-2}(x-1) + \dots + (x-1)^{k-1} + (-1)^{k+1}(x-1)), \\ \psi_k(x) &= \frac{1}{x} (x^{k-1} + x^{k-2}(x-1) + \dots + (x-1)^{k-1} + (-1)^k). \end{aligned}$$

For each  $i \in \{0, \dots, k-1\}$  with  $i \neq k-2$ , we have  $x^i(x-1)^{k-1-i} \leq x^{k-1}$  while  $x^{k-2}(x-1) + 1 \leq x^{k-2}(x-1) + (x-1) \leq x^{k-1}$ , for all  $x \geq 2$ . The result now follows.

c) For any  $n \geq 0$ , the set  $\mathcal{L}_k(n)$  of solutions in  $\{0, \dots, p-1\}$  of the linear congruence (4) has size  $p^{k-1}$ . Moreover, the solutions of (4) with no coordinate being zero is the complement of the solutions described in a), with respect to  $\mathcal{L}_k(n)$ , that is, the number of solutions of (4) with no coordinate being zero is

$$\begin{aligned} S_k(p) &= p^{k-1} - \phi_k(p) & \text{if } p \mid n; \\ W_k(p) &= p^{k-1} - \psi_k(p) & \text{if } p \nmid n. \end{aligned}$$

d) The solutions of (4) with at most one coordinate being zero is the disjoint union of the solutions of (4) with no coordinate being zero and the solutions of (4)

with exactly one coordinate being zero. Thus, by c), we get that the number of solutions of (4) with at most one coordinate being zero is

$$\begin{aligned} S_k(p) + kS_{k-1}(p) &= \eta_k(p) & \text{if } p \mid n; \\ W_k(p) + kW_{k-1}(p) &= \delta_k(p) & \text{if } p \nmid n. \end{aligned}$$

e) For any  $n \geq 0$ , the linear congruence (4) has exactly  $p^{k-1}$  solutions in  $\{0, \dots, p-1\}$ , which are obtained choosing freely the values of  $k-1$  variables  $y_i^*$  and computing the last one. Moreover, for all  $p$ , by a) and d), we can interpret  $\psi_k(p)$ ,  $\delta_k(p)$  as counting the number of particular solutions of (4) with  $n = 1$  and  $\phi_k(p)$ ,  $\eta_k(p)$  as counting the number of particular solutions of (4) with  $n = 0$ . This gives for any  $k \geq 1$ ,

$$\max\{\psi_k(p), \delta_k(p), \phi_k(p), \eta_k(p)\} \leq p^{k-1}.$$

Assume now  $k \geq 3$  and observe that, by d),  $\delta_k(p)$  is the number of solutions of (4) with at most one  $y_j^* = 0$  and with  $n = 1$ . Among these solutions we find those obtained by selecting arbitrarily  $y_2^*, \dots, y_k^* \in \{1, \dots, p-1\}$  and determining the corresponding  $y_1^*$ , which says  $\delta_k(p) \geq (p-1)^{k-1}$ . On the other hand, since we cannot assign 0 in two of the  $k-1 \geq 2$  variables  $y_2^*, \dots, y_k^*$ , we have  $\delta_k(p) < p^{k-1}$ .

f) To get

$$\min\{\psi_k(p), \delta_k(p), \phi_k(p), \eta_k(p)\} \geq 1,$$

observe that there exists at least one solution for those equations when  $k \geq 2$ .

g) We begin showing that, for all primes  $p$  and  $k \geq 3$ , the inequality

$$(5) \quad \max\{p^{k-1} - \delta_k(p), p^{k-1} - \eta_k(p)\} \leq k2^{k-1}p^{k-3}$$

holds. By expanding the terms in  $\delta_k(p)$ , we get

$$\begin{aligned} \delta_k(p) &= \sum_{i=1}^k (-1)^{k-i} \binom{k}{i} p^{i-1} + k \sum_{i=1}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} p^{i-1} \\ &= p^{k-1} - \sum_{i=1}^{k-2} (-1)^{k-i} \left[ k \binom{k-1}{i} - \binom{k}{i} \right] p^{i-1}. \end{aligned}$$

Hence, by e),

$$(6) \quad 0 \leq p^{k-1} - \delta_k(p) = \sum_{i=1}^{k-2} (-1)^{k-i} \left[ k \binom{k-1}{i} - \binom{k}{i} \right] p^{i-1}$$

and

$$(7) \quad 0 \leq p^{k-1} - \eta_k(p) = p^{k-1} - \delta_k(p) + (-1)^k(k-1).$$

Note that since

$$k \binom{k-1}{i} = (k-i) \binom{k}{i} \geq \binom{k}{i} \quad \text{for all } i \in \{1, \dots, k-2\},$$

we have

$$k \binom{k-1}{i} - \binom{k}{i} \geq 0.$$

If  $k$  is odd, we have from (6)

$$p^{k-1} - \delta_k(p) \leq \sum_{i=1}^{k-2} k \binom{k-1}{i} p^{i-1} \leq k2^{k-1}p^{k-3}.$$

By (7), we have  $p^{k-1} - \eta_k(p) \leq p^{k-1} - \delta_k(p)$ , and so the same inequality holds for  $p^{k-1} - \eta_k(p)$ .

If  $k$  is even, then  $k-1$  is odd, so that the term corresponding to the choice  $i=1$  in the sum in (6) is negative; moreover, since  $k-2 \geq 2$ , there is at least another term in the sum in (6). It follows, by (7), that

$$\begin{aligned} p^{k-1} - \eta_k(p) &= p^{k-1} - \delta_k(p) + k - 1 \leq \sum_{i=2}^{k-2} k \binom{k-1}{i} p^{k-3} + k - 1 \\ &= k(2^{k-1} - k - 1)p^{k-3} + k - 1 \leq k2^{k-1}p^{k-3}, \end{aligned}$$

because  $k(k+1)p^{k-3} \geq k-1$  for any  $k \geq 3$ . The same conclusion follows also for  $p^{k-1} - \delta_k(p)$  since  $p^{k-1} - \delta_k(p) \leq p^{k-1} - \eta_k(p)$ . So, we have proved (5).

Therefore we can write

$$\frac{\delta_k(p)}{p^{k-1}} = 1 - \frac{\delta'_k(p)}{p^2}, \quad \text{with } 0 \leq \delta'_k(p) \leq k2^{k-1},$$

as well as

$$\frac{\eta_k(p)}{p^{k-1}} = 1 - \frac{\eta'_k(p)}{p^2}, \quad \text{with } 0 \leq \eta'_k(p) \leq k2^{k-1}.$$

Thus,

$$\frac{p^{k-1}}{\delta_k(p)} = \frac{1}{1 - \delta'_k(p)/p^2} \leq \frac{1}{1 - k2^{k-1}/p^2} \leq 1 + \frac{k2^k}{p^2}$$

for  $p^2 \geq k2^k$ , using the fact that

$$\frac{1}{1-y} \leq 1 + 2y, \quad \text{for all } 0 \leq y \leq 1/2.$$

The same argument applies to  $\eta_k(p)$  and gives

$$\max \left\{ \frac{p^{k-1}}{\eta_k(p)}, \frac{p^{k-1}}{\delta_k(p)} \right\} \leq 1 + \frac{k2^k}{p^2}$$

for any  $k \geq 3$  and any prime  $p \geq \sqrt{k2^k}$ . □

Using Lemma 8, we are now ready to prove Theorems 3 and 5.

*Proof of Theorem 3.* Let  $k \geq 2$  and consider

$$C_k = \prod_p \left( 1 - \frac{\psi_k(p)}{p^k} \right) = \exp \left\{ - \sum_p - \log \left( 1 - \frac{\psi_k(p)}{p^k} \right) \right\}.$$

By Lemma 8 e) and f), we have

$$1 \leq \psi_k(p) \leq p^{k-1}$$

and therefore  $1/p^k \leq \psi_k(p)/p^k \leq 1/p$ , which shows that  $0 < 1 - \psi_k(p)/p^k < 1$ . So the series

$$(8) \quad \sum_p - \log \left( 1 - \frac{\psi_k(p)}{p^k} \right)$$



has positive terms and  $C_k$  is a real number in  $[0, 1)$ . We will show that  $C_k \neq 0$  by observing that the series (8) converges. To do that, we expand

$$\begin{aligned} 1 - \frac{\psi_k(p)}{p^k} &= 1 - \frac{p^k - (p-1)^k + (-1)^k}{p} \\ &= 1 - \frac{k}{p^2} - \sum_{i=2}^{k-1} (-1)^{i+1} \binom{k}{i} \frac{1}{p^{i+1}} \\ &= 1 - \frac{k}{p^2} + o\left(\frac{1}{p^2}\right) \quad \text{as } p \rightarrow \infty, \end{aligned}$$

and therefore we obtain

$$-\log\left(1 - \frac{\psi_k(p)}{p^k}\right) \sim \frac{k}{p^2} \quad \text{as } p \rightarrow \infty.$$

Since

$$\sum_p \frac{k}{p^2} < \sum_{n=1}^{\infty} \frac{k}{n^2} = \frac{k\pi^2}{6}$$

converges, also the series (8) converges.

We now turn to the inequalities involving the functions

$$f_k(n) = \prod_{p|n} \left(1 + \frac{(-1)^k}{p^k - \psi_k(p)}\right).$$

By Lemma 8 e), we have

$$p^k - \psi_k(p) \geq p^k - p^{k-1} > 0$$

and thus for any  $n \in \mathbb{N}$  we get  $f_k(n) < 1$  if  $k$  is odd, and  $f_k(n) > 1$  if  $k$  is even. Observe also that the function  $p^k - p^{k-1}$  is increasing in  $k$ . To find an upper bound when  $k$  is even, note that  $p^k - p^{k-1} \leq p^2 - p$  and thus

$$f_k(n) \leq \prod_p \left(1 + \frac{1}{p^2 - p}\right).$$

It follows that

$$\begin{aligned} \log(f_k(n)) &\leq \sum_p \log\left(1 + \frac{1}{p^2 - p}\right) \\ &= \sum_{p \leq 1000} \log\left(1 + \frac{1}{p^2 - p}\right) + \sum_{p > 1000} \log\left(1 + \frac{1}{p^2 - p}\right) \\ &< 0.665 + \sum_{n \geq 1001} \frac{1}{n^2 - n} = 0.665 + 0.001 = 0.666, \end{aligned}$$

which gives

$$f_k(n) < e^{0.666} < 2.$$

Finally, we find a lower bound when  $k$  is odd starting from  $p^k - p^{k-1} \leq p^3 - p^2$ , which immediately gives

$$f_k(n) \geq \prod_p \left(1 - \frac{1}{p^3 - p^2}\right).$$

It follows that

$$\log(f_k(n)) \geq - \sum_p \log \left( \left( 1 - \frac{1}{p^3 - p^2} \right)^{-1} \right),$$

Since

$$\begin{aligned} \sum_p \log \left( \left( 1 - \frac{1}{p^3 - p^2} \right)^{-1} \right) &= \sum_p \log \left( 1 + \frac{1}{p^3 - p^2 - 1} \right) \\ &= \sum_{p \leq 1001} \log \left( 1 + \frac{1}{p^3 - p^2 - 1} \right) \\ &\quad + \sum_{p > 1001} \log \left( 1 + \frac{1}{p^3 - p^2 - 1} \right) \\ &< 0.361 + \sum_{p > 1001} \frac{1}{p^3 - p^2 - 1} \\ &< 0.361 + \sum_{p > 1001} \frac{1}{p(p-1)(p-2)} \\ &< 0.361 + \sum_{n \geq 1002} \left( \frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2n} \right) \\ &< 0.361 + 0.0005 = 0.3615, \end{aligned}$$

it follows that

$$f_k(n) \geq e^{-0.3615} > \frac{2}{3}.$$

□

*Proof of Theorem 5.* Let  $k \geq 3$ . From Lemma 8 e) and f), we have

$$1 \leq \delta_k(p) < p^{k-1} \quad \text{so that} \quad 0 < \frac{\delta_k(p)}{p^{k-1}} < 1,$$

and

$$D_k = \prod_p \frac{\delta_k(p)}{p^{k-1}} = \exp \left\{ - \sum_p \log \left( \frac{p^{k-1}}{\delta_k(p)} \right) \right\},$$

where the series

$$(9) \quad \sum_p \log \left( \frac{p^{k-1}}{\delta_k(p)} \right)$$

has positive terms. This gives immediately that  $D_k \in [0, 1)$  and we need only to show that  $D_k \neq 0$ . To do that we prove that the series (9) converges.

For  $p \geq \sqrt{k2^k}$ , Lemma 8 g) gives

$$\frac{p^{k-1}}{\delta_k(p)} \leq 1 + \frac{k2^k}{p^2} \quad \text{and therefore} \quad \log \left( \frac{p^{k-1}}{\delta_k(p)} \right) \leq \frac{k2^k}{p^2}.$$

Thus,

$$\sum_p \log \left( \frac{p^{k-1}}{\delta_k(p)} \right) \leq \sum_{p < \sqrt{k2^k}} \log \left( \frac{p^{k-1}}{\delta_k(p)} \right) + k2^k \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where the last series converges.

We now turn to the inequalities involving the functions  $g_k(n)$  for  $k \geq 3$ . First of all, observe that  $g_k(n) < 1$  if  $k$  is even, and  $g_k(n) > 1$  if  $k$  is odd, because  $(k-1)/\delta_k(p) > 0$ . To get some bounds for  $g_k(n)$ , we begin computing:

$$\delta_k(2) = \begin{cases} 1 & \text{if } k \text{ is odd,} \\ k & \text{if } k \text{ is even,} \end{cases}$$

and

$$\delta_3(p) = p^2 - 3, \quad \delta_4(p) = p^3 - 6p + 8.$$

Recall that, by Lemma 8 e), for any  $k \geq 3$ , we have  $\delta_k(p) \geq (p-1)^{k-1}$ . Let  $k$  be odd. For any  $p \geq 3$ , we have

$$\frac{k-1}{\delta_k(p)} \leq \frac{2}{\delta_3(p)}, \quad \text{that is} \quad \delta_k(p) \geq \frac{(k-1)(p^2-3)}{2}$$

(this is trivial when  $k=3$  and, for  $k \geq 5$ , we have  $(p-1)^{k-1} \geq (k-1)(p^2-3)/2$ ). Similarly if  $k$  is even, for any  $p \geq 3$ , we have

$$\frac{k-1}{\delta_k(p)} \leq \frac{3}{\delta_4(p)}, \quad \text{that is} \quad \delta_k(p) \geq \frac{(k-1)(p^3-6p+8)}{3}$$

(this is trivial when  $k=4$  and, for  $k \geq 6$ , we have  $(p-1)^{k-1} \geq (k-1)(p^3-6p+8)/3$ ). It follows that when  $k$  is odd

$$g_k(n) = \prod_{p|n} \left( 1 + \frac{k-1}{\delta_k(p)} \right) \leq \begin{cases} k \prod_{p>2} \left( 1 + \frac{2}{p^2-3} \right) & \text{if } n \text{ is even,} \\ \prod_{p>2} \left( 1 + \frac{2}{p^2-3} \right) & \text{if } n \text{ is odd.} \end{cases}$$

Similarly, for  $k$  even, we obtain

$$g_k(n) = \prod_{p|n} \left( 1 - \frac{k-1}{\delta_k(p)} \right) \geq \begin{cases} \frac{1}{k} \prod_{p>2} \left( 1 - \frac{3}{p^3-6p+8} \right) & \text{if } n \text{ is even,} \\ \prod_{p>2} \left( 1 - \frac{3}{p^3-6p+8} \right) & \text{if } n \text{ is odd.} \end{cases}$$

It remains to give estimates for the numbers

$$a = \prod_{p>2} \left( 1 + \frac{2}{p^2-3} \right) \quad \text{and} \quad b = \prod_{p>2} \left( 1 - \frac{3}{p^3-6p+8} \right).$$

We have

$$\begin{aligned} \log a &= \sum_{p \geq 3} \log \left( 1 + \frac{2}{p^2-3} \right) \leq 2 \sum_{p \geq 3} \frac{1}{p^2-3} \\ &\leq 2 \left[ \sum_{p=3}^{11} \frac{1}{p^2-3} + \frac{13^2}{13^2-3} \int_{13}^{\infty} \frac{dx}{x^2} \right] < \log 2, \end{aligned}$$

and so  $a < 2$ . Finally, observe that

$$\log b = - \sum_{p \geq 3} \log \left( 1 + \frac{3}{p^3-6p+5} \right)$$

and that

$$\begin{aligned} \sum_{p \geq 3} \log \left( 1 + \frac{3}{p^3 - 6p + 5} \right) &\leq \sum_{p \geq 3} \frac{3}{p^3 - 6p + 5} \leq \frac{3}{14} + 4 \sum_{p \geq 5} \frac{1}{p^3} \\ &\leq \frac{3}{14} + 4 \left[ \frac{1}{125} + \int_5^\infty \frac{dx}{x^3} \right] < \log 2. \end{aligned}$$

It follows that  $b > 1/2$ . □

#### 4. THE COEFFICIENTS $K_{k,d}(n)$ AND THE PROOF OF THEOREM 2

Now we are ready to compute  $K_{k,d}(n)$ . We will show in Proposition 11 that the leading term of  $K_{k,d}(n)$  is  $(n/d)^k/k!$  multiplied by the correction factor defined below.

**Definition 9.** For positive integers  $k$  and  $n$ , write

$$\Theta_{k,n}(d) = \prod_{p|d, p|n} \phi_k(p) \prod_{p|d, p \nmid n} \psi_k(p),$$

where for  $d = 1$  the empty product is interpreted 1.

The function  $\Theta_{k,n}(d)$  of the square-free number  $d$  is multiplicative; that is, if  $d_1$  and  $d_2$  are coprime square-free numbers, then  $\Theta_{k,n}(d_1 d_2) = \Theta_{k,n}(d_1) \Theta_{k,n}(d_2)$ .

Note also that, by Lemma 8 b) and f), we have

$$(10) \quad 1 \leq \Theta_{k,n}(d) \leq k^{\omega(d)} d^{k-2} \quad \text{for all } k \geq 2,$$

where, for a positive integer  $m$ ,  $\omega(m)$  denotes the number of distinct prime divisors of  $m$ .

Here and in the next section, to go straight on into computations, we need also this technical lemma.

**Lemma 10.** Let  $x, y$  and  $c \geq 1$  be real numbers and  $k \in \mathbb{N}$ . If  $|x - y| \leq ck$ , then

$$|x^k - y^k| < \frac{k! e^{1/c} (ce)^k}{\sqrt{2\pi k}} y^{k-1}.$$

*Proof.* Let  $x = y + \theta ck$ . We then have

$$x^k = y^k + \sum_{i=0}^{k-1} \binom{k}{i} y^i (ck\theta)^{k-i}$$

and

$$|x^k - y^k| \leq y^{k-1} \sum_{i=0}^{k-1} \binom{k}{i} (ck)^{k-i} < y^{k-1} \sum_{i=0}^k \binom{k}{i} (ck)^{k-i} = y^{k-1} (ck + 1)^k.$$

As the exponential function with base greater than 1 is increasing, we obtain

$$(ck + 1)^k = (ck)^k \left[ \left( 1 + \frac{1}{ck} \right)^{ck} \right]^{1/c} < (ck)^k e^{1/c}.$$

By Stirling's formula,

$$(11) \quad k! > \left( \frac{k}{e} \right)^k \sqrt{2\pi k} \quad \text{and} \quad k^k < \frac{k! e^k}{\sqrt{2\pi k}}.$$

Inserting the inequality from the right-hand side of (11), we get the desired conclusion.  $\square$

**Proposition 11.** *Let  $k \geq 1$  and  $1 \leq d \leq n$ . Then*

$$\left| K_{k,d}(n) - \Theta_{k,n}(d) \frac{(n/d)^k}{k!} \right| \leq \Theta_{k,n}(d) \left( k + \frac{e^{k+1}}{\sqrt{2\pi k}} \right) (n/d)^{k-1}.$$

*Proof.* If  $d = 1$ , we have

$$K_{k,1}(n) = K_k(n) = \frac{n^k}{k!} + \theta_1 k n^{k-1} = \frac{n^k}{k!} + \theta_2 \left( k + \frac{e^{k+1}}{\sqrt{2\pi k}} \right) n^{k-1}.$$

Suppose that  $d > 1$ . By definition,  $K_{k,d} = \#\mathcal{K}_{k,d}(n)$  and the elements of  $\mathcal{K}_{k,d}(n)$  are the solutions  $(x, y_1, \dots, y_k)$  of the equation

$$(12) \quad n = x + \sum_{j=1}^k y_j, \quad \text{for which } d \text{ divides } x \text{ and } y_1 \cdots y_k.$$

Write  $x = dX$ , and  $y_j = y_j^* + dY_j$  with  $X > 0$ ,  $Y_j \geq 0$  and  $y_j^* \in \{0, \dots, d-1\}$ , for each  $j \in \{1, \dots, k\}$ . Note that, as  $d$  divides  $\prod_{j=1}^k y_j$ , for each prime factor  $p$  of  $d$ , there exists at least an index  $j_p \in \{1, \dots, k\}$  with  $p \mid y_{j_p}^*$ . Clearly,  $(y_1^*, \dots, y_k^*)$  and  $(X, Y_1, \dots, Y_k)$  are uniquely determined by  $(x, y_1, \dots, y_k)$ , and similarly, the vector  $(x, y_1, \dots, y_k)$  is uniquely determined by both  $(y_1^*, \dots, y_k^*)$  and  $(X, Y_1, \dots, Y_k)$ .

We now determine the number of possible tuples  $(y_1^*, \dots, y_k^*)$ . Reducing (12) modulo  $d$ , we get

$$(13) \quad y_1^* + \cdots + y_k^* \equiv n \pmod{d}, \quad \text{with } y_1^* \cdots y_k^* \equiv 0 \pmod{d}.$$

Reducing congruence (13) further modulo  $p$ , where  $p$  is an arbitrary prime factor of  $d$ , we get a solution to the equation

$$(14) \quad y_{1,p}^* + \cdots + y_{k,p}^* \equiv n \pmod{p},$$

with  $y_{j_p,p}^* \equiv 0 \pmod{p}$  for at least one  $j_p \in \{1, \dots, k\}$ .

This shows that  $(y_1^*, \dots, y_k^*)$  determines a solution  $(y_{1,p}^*, \dots, y_{k,p}^*)$  of (14), for each prime factor  $p$  of  $d$ .

Conversely, for each prime factor  $p$  of  $d$ , let  $(y_{1,p}^*, \dots, y_{k,p}^*) \in \mathbb{Z}_p^k$  be a solution of (14). Fix  $j \in \{1, \dots, n\}$ , consider the system  $y_j^* \equiv y_{j,p}^* \pmod{p}$  for  $p \mid d$  and apply the Chinese remainder theorem to find a unique solution modulo  $d$ . Looking at the equation related to  $p$  in each system, we have  $y_1^* + \cdots + y_k^* \equiv n \pmod{p}$  for all  $p \mid d$  and, since  $d$  is square-free, it follows that  $(y_1^*, \dots, y_k^*) \in \mathbb{Z}_d^k$  is a solution of (13). Note that  $y_1^* \cdots y_k^* \equiv 0 \pmod{d}$  is a consequence of  $y_{j_p,p}^* \equiv 0 \pmod{p}$  for at least one  $j_p \in \{1, \dots, k\}$ , because this implies that  $p$  divides  $y_{j_p}^*$  and consequently  $d = \prod_{p \mid d} p$  divides  $y_1^* \cdots y_k^*$ .

Now, by Lemma 8 a), the number of solutions of (14) is either  $\phi_k(p)$  (if  $p \mid n$ ) or  $\psi_k(p)$  (if  $p \nmid n$ ). Hence, the number of possibilities for  $(y_1^*, \dots, y_k^*)$  is

$$\prod_{p \mid d, p \mid n} \phi_k(p) \prod_{p \mid d, p \nmid n} \psi_k(p) = \Theta_{k,n}(d).$$

Fix  $(y_1^*, \dots, y_k^*)$  and let us determine the number of possible tuples  $(X, Y_1, \dots, Y_k)$ . From (12), we get the equation

$$(15) \quad X + \sum_{j=1}^k Y_j = \frac{n - \sum_{j=1}^k y_j^*}{d},$$

where the right-hand side is an integer. Recalling that  $X$  and  $Y_j$  are non-negative, it follows that the number of solutions of (15), with respect to the natural number  $m := (n - \sum_{j=1}^k y_j^*)/d$ , is between the number of  $(k+1)$ -compositions of  $m$  and the number of generalized  $(k+1)$ -compositions of  $m$ . Thus, by (1), its size is

$$\frac{m^k}{k!} + \theta_3 k m^{k-1}.$$

Since  $y_j^* \in \{0, \dots, d-1\}$ , we have

$$0 \leq \sum_{j=1}^k \frac{y_j^*}{d} \leq \left(1 - \frac{1}{d}\right) k \leq k.$$

So,  $|m - n/d| \leq k$ . Therefore, applying Lemma 10 with  $c = 1$ , we get

$$\left| \frac{m^k}{k!} - \frac{(n/d)^k}{k!} \right| \leq \frac{e^{k+1}}{\sqrt{2\pi k}} (n/d)^{k-1}.$$

To estimate  $km^{k-1}$ , we note that  $m \leq n/d$  gives  $km^{k-1} \leq k(n/d)^{k-1}$ . Thus, for a fixed  $(y_1^*, \dots, y_k^*)$ , the number of acceptable integer solutions  $(X, Y_1, \dots, Y_k)$  to equation (15) is

$$\frac{(n/d)^k}{k!} + \theta_4 \left[ \left( k + \frac{e^{k+1}}{\sqrt{2\pi k}} \right) (n/d)^{k-1} \right].$$

Except for the value of  $\theta_4$ , this does not depend on  $(y_1^*, \dots, y_k^*)$ . Summing up the above expression over the possible  $(y_1^*, \dots, y_k^*) \in \mathbb{Z}_d^k$ , we get the desired result.  $\square$

The following elementary observation will be used in the proof of Theorem 2.

**Lemma 12.** *For  $k \geq 0$ , let*

$$I_k(x) = \int_x^\infty \frac{(\log t)^k}{t^2} dt$$

*as a function in the real variable  $x > 0$ . Then*

$$I_k(x) \leq 2k! \frac{(\log x)^k}{x} \quad \text{for all} \quad x \geq e^{4/3}.$$

*Proof.* For  $k = 0$ , we have

$$I_0(x) = \int_x^\infty \frac{dt}{t^2} = \frac{1}{x}$$

and the lemma is trivial. For  $k \geq 1$ , we have

$$\begin{aligned} I_{k-1}(x) &= \int_x^\infty \frac{(\log t)^{k-1}}{t} \frac{dt}{t} = \frac{(\log t)^k}{k} \frac{1}{t} \Big|_x^\infty + \int_x^\infty \frac{\log t^k}{k} \frac{1}{t^2} dt \\ &= -\frac{(\log x)^k}{kx} + \frac{1}{k} I_k(x) \end{aligned}$$

which gives

$$I_k(x) = \frac{(\log x)^k}{x} + k I_{k-1}(x).$$

Using this relation, the lemma follows by induction on  $k \geq 1$ . In fact,

$$I_1(x) = \frac{\log x}{x} + \frac{1}{x} \leq \frac{2 \log x}{x} \quad \text{for any } x \geq e,$$

and so, in particular, for all  $x \geq e^{4/3}$ . Moreover, by the inductive hypothesis,

$$I_{k+1}(x) = \frac{(\log x)^{k+1}}{x} + (k+1)I_k(x) \leq \frac{(\log x)^{k+1}}{x} \left[ 1 + \frac{2(k+1)!}{\log x} \right]$$

and, for  $x \geq e^{4/3}$ , we get

$$1 + \frac{2(k+1)!}{\log x} \leq 1 + \frac{3(k+1)!}{2} \leq 2(k+1)!$$

for all  $k \geq 1$ . □

*Proof of Theorem 2.* Due to the cases discussed in the Introduction, we can assume that  $k \geq 2$  and  $n \geq 4$ . We begin by applying (2) together with Proposition 11

$$\begin{aligned} A_k(n) &= \sum_{1 \leq d \leq n} \mu(d) K_{k,d}(n) \\ &= \sum_{1 \leq d \leq n} \mu(d) \Theta_{k,n}(d) \left( \frac{(n/d)^k}{k!} + \theta_d \left( k + \frac{e^{k+1}}{\sqrt{2\pi k}} \right) (n/d)^{k-1} \right) \\ (16) \quad &= M + E \end{aligned}$$

where

$$M = \sum_{1 \leq d \leq n} \mu(d) \Theta_{k,n}(d) \frac{(n/d)^k}{k!}$$

is the main term and

$$E = \sum_{1 \leq d \leq n} \mu(d) \Theta_{k,n}(d) \theta_d \left( k + \frac{e^{k+1}}{\sqrt{2\pi k}} \right) (n/d)^{k-1}$$

is the error term.

Thus, by (10), we get

$$\begin{aligned} (17) \quad E &= \theta_1 \left( k + \frac{e^{k+1}}{\sqrt{2\pi k}} \right) n^{k-1} \sum_{1 \leq d \leq n} \frac{\Theta_{k,n}(d)}{d^{k-1}} \\ &= \theta_2 \left( k + \frac{e^{k+1}}{\sqrt{2\pi k}} \right) n^{k-1} \sum_{1 \leq d \leq n} \frac{k^{\omega(d)}}{d}. \end{aligned}$$

We want to find a better estimate for  $E$  through an estimate for the function:

$$\Omega_k(x) = \sum_{1 \leq d \leq x} \frac{k^{\omega(d)}}{d},$$

defined for any real number  $x \geq 1$ . From [13, (3.20)], we have

$$\sum_{p \leq x} \frac{1}{p} \leq \log \log x + B + \frac{1}{(\log x)^2},$$

for each real number  $x > 1$ , where  $B$  is the Mertens' constant [13, (2.10)]. As  $B \leq 0.27$ , we have, in particular, that

$$\sum_{p \leq x} \frac{1}{p} \leq \log \log x + 1, \quad \text{for any } x \geq 4.$$

Hence, for any real number  $x \geq 4$ , we have also

$$\begin{aligned} \Omega_k(x) &\leq \prod_{p \leq x} \left(1 + \frac{k}{p}\right) \leq \exp \left( \sum_{p \leq x} \frac{k}{p} \right) \\ (18) \quad &\leq \exp(k(\log \log x + 1)) = (e \log x)^k. \end{aligned}$$

Using (17), we find

$$(19) \quad E = \theta_3 \left( k + \frac{e^{k+1}}{\sqrt{2\pi k}} \right) (e \log n)^k n^{k-1}.$$

We now look at the main term  $M$ . We have

$$(20) \quad M = \frac{n^k}{k!} \sum_{d \geq 1} \mu(d) \frac{\Theta_{k,n}(d)}{d^k} - \frac{n^k}{k!} \sum_{d > n} \mu(d) \frac{\Theta_{k,n}(d)}{d^k} = \frac{n^k}{k!} (M_1 - M_2).$$

We first compute

$$M_1 = \sum_{d \geq 1} \mu(d) \frac{\Theta_{k,n}(d)}{d^k}$$

and then we estimate

$$M_2 = \sum_{d > n} \mu(d) \frac{\Theta_{k,n}(d)}{d^k}.$$

For  $M_1$  we have, by the multiplicativity of  $\Theta_{k,n}(d)$  as a function of  $d$ ,

$$\begin{aligned} M_1 &= \prod_{p|n} \left(1 - \frac{\phi_k(p)}{p^k}\right) \prod_{p \nmid n} \left(1 - \frac{\psi_k(p)}{p^k}\right) \\ &= \prod_{p|n} \left(1 - \frac{\phi_k(p)}{p^k}\right) \left(1 - \frac{\psi_k(p)}{p^k}\right)^{-1} \prod_p \left(1 - \frac{\psi_k(p)}{p^k}\right) \\ (21) \quad &= C_k f_k(n), \end{aligned}$$

where the last equality arises upon observing that, by (3),

$$\left(1 - \frac{\phi_k(p)}{p^k}\right) \left(1 - \frac{\psi_k(p)}{p^k}\right)^{-1} = \frac{p^k - \phi_k(p)}{p^k - \psi_k(p)} = 1 + \frac{(-1)^k}{p^k - \psi_k(p)}.$$

For  $M_2$ , we use (10) to conclude that

$$|M_2| \leq \sum_{d > n} \frac{k^{\omega(d)}}{d^2}.$$

By the Euler summation formula on  $\Omega_k(x)$  (see Theorem 3.1 in [1]), we get

$$\sum_{x < d \leq X} \frac{k^{\omega(d)}}{d^2} = \int_x^X \frac{(\Omega_k(t))'}{t} dt = \frac{\Omega_k(t)}{t} \Big|_x^X + \int_x^X \frac{\Omega_k(t)}{t^2} dt,$$



for any  $X > x$ . Since (18) implies that  $\Omega_k(t) = O((e \log t)^k)$ , taking  $X \rightarrow \infty$ , the first summand is equal to  $-\Omega_k(x)/x$  and, in particular, is negative. Therefore, using again (18), we deduce that for  $x \geq 4$ :

$$\sum_{x < d} \frac{k^{\omega(d)}}{d^2} \leq \int_x^\infty \frac{\Omega_k(t)}{t^2} dt \leq e^k \int_x^\infty \frac{(\log t)^k}{t^2} dt.$$

Hence, from Lemma 12, we obtain

$$(22) \quad |M_2| \leq \frac{2k!e^k(\log n)^k}{n}.$$

Since, for  $k \geq 2$ , we have

$$\left(k + \frac{e^{k+1}}{\sqrt{2\pi k}}\right) e^k + 2e^k = \left(k + 2 + \frac{e^{k+1}}{\sqrt{2\pi k}}\right) e^k < \frac{(2+e)e^{2k}}{\sqrt{2\pi k}},$$

the desired conclusion follows finally from (16), (19), (20), (21) and (22).  $\square$

## 5. PROOF OF THEOREM 4

We start with two definitions and a proposition which play a role similar to Definitions 6, 9 and Proposition 11.

**Definition 13.** Given positive integers  $k$  and  $n$ , write  $\mathcal{B}_{k,d}(n)$  for the set of  $k$ -compositions  $(x_1, \dots, x_k)$  of  $n$  such that  $\gcd(x_i, x_j)$  is coprime to  $d$ , for every distinct  $i, j \in \{1, \dots, k\}$ , that is,

$$\mathcal{B}_{k,d}(n) = \{(x_1, \dots, x_k) \in \mathbb{N}^k : n = x_1 + \dots + x_k \text{ and } p \nmid \gcd(x_i, x_j), \text{ for each prime } p \mid d \text{ and for distinct } i, j \in \{1, \dots, k\}\}.$$

and put  $B_{k,d}(n) = \#\mathcal{B}_{k,d}(n)$ .

Clearly  $\mathcal{B}_{k,1}(n) = \mathcal{K}_{k-1}(n)$  and  $\mathcal{B}_k(n) \subseteq \mathcal{B}_{k,d}(n)$ .

**Definition 14.** For positive integers  $k$  and  $n$ , write

$$\Xi_{k,n}(d) = \prod_{p \mid d, p \mid n} \eta_k(p) \prod_{p \mid d, p \nmid n} \delta_k(p),$$

where for  $d = 1$  the empty product is taken to be 1.

Note that, by Lemma 8 e) and f), we have

$$(23) \quad 1 \leq \Xi_{k,n}(d) \leq d^{k-1}.$$

**Proposition 15.** Let  $k \geq 3$  and  $1 \leq d \leq n$ . Then

$$\left| B_{k,d}(n) - \Xi_{k,n}(d) \frac{(n/d)^{k-1}}{(k-1)!} \right| \leq \Xi_{k,n}(d) \left( k-1 + \frac{e^{2/3}(3e/2)^{k-1}}{\sqrt{2\pi(k-1)}} \right) n^{k-2}d.$$

*Proof.* The proof of this proposition is very similar to the proof of Proposition 11. For  $d = 1$  the statement is trivial because, by (1),

$$B_{k,1}(n) = K_{k-1}(n) = \frac{n^{k-1}}{(k-1)!} + \theta_1(k-1)n^{k-2}.$$

Let  $d > 1$ . By Definition 13, the elements  $(x_1, \dots, x_k)$  of  $\mathcal{B}_{k,d}(n)$  are the solutions of the equation

$$(24) \quad n = x_1 + \dots + x_k \text{ for which } d \text{ is coprime to } \gcd(x_i, x_j), \text{ for every } i < j.$$

Write  $x_i = dX_i + x_i^*$  with  $X_i \geq 0$  and  $x_i^* \in \mathbb{Z}_d$ , for each  $i \in \{1, \dots, k\}$ . First we examine the possible tuples  $(x_1^*, \dots, x_k^*) \in \mathbb{Z}_d^k$  which can arise.

Reducing (24) modulo  $d$ , we get

$$(25) \quad x_1^* + \dots + x_k^* \equiv n \pmod{d} \quad \text{with } p \nmid \gcd(x_i^*, x_j^*), \text{ for each } p \mid d \text{ and } i \neq j,$$

and thus  $(x_1^*, \dots, x_k^*)$  belongs to the set

$$\mathcal{S} = \{(x_1^*, \dots, x_k^*) \in \mathbb{Z}_d^k : x_1^* + \dots + x_k^* \equiv n \pmod{d} \text{ with } p \nmid \gcd(x_i^*, x_j^*), \\ \text{for each } p \mid d \text{ and } i \neq j\}.$$

Reducing (25) further modulo  $p$ , where  $p$  is an arbitrary prime factor of  $d$ , we get a unique solution to the equation

$$(26) \quad x_{1,p}^* + \dots + x_{k,p}^* \equiv n \pmod{p} \quad \text{with } x_{i,p}^* \in \mathbb{Z}_p \text{ and } x_{i,p}^* = 0 \text{ for at most one } i.$$

Note that, by Lemma 8 d), the number of solutions of (26) is either  $\eta_k(p)$  (if  $p \mid n$ ) or  $\delta_k(p)$  (if  $p \nmid n$ ). Moreover, if we consider a solution  $(x_{1,p}^*, \dots, x_{k,p}^*) \in \mathbb{Z}_p^k$  of (26) for any prime factor  $p$  of  $d$  and apply the Chinese remainder theorem in each one of the  $k$  coordinates, we get a unique solution  $(x_1^*, \dots, x_k^*) \in \mathbb{Z}_d^k$  of (25) with  $x_i^* \equiv x_{i,p}^* \pmod{p}$ , for all  $i \in \{1, \dots, k\}$  and all  $p \mid d$ . Hence,

$$(27) \quad \#\mathcal{S} = \prod_{p \mid d, p \mid n} \eta_k(p) \prod_{p \mid d, p \nmid n} \delta_k(p) = \Xi_{k,n}(d).$$

Now we fix  $x^* = (x_1^*, \dots, x_k^*) \in \mathcal{S}$  and we determine the possible tuples  $X = (X_1, \dots, X_k)$  which are compatible with (24), that is, the solutions  $\mathcal{T}_{x^*}$  of

$$X_1 + \dots + X_k = \frac{(n - \sum_{i=1}^k x_i^*)}{d},$$

where the right-hand side  $m = (n - \sum_{i=1}^k x_i^*)/d$  is an integer. Clearly there is a bijection between  $\mathcal{B}_{k,d}(n)$  and

$$\{(x^*, X) : x^* \in \mathcal{S}, X \in \mathcal{T}_{x^*}\}.$$

So, to estimate  $B_{k,d}(n)$  we just need to estimate  $\#\mathcal{T}_{x^*}$ , for every  $x^* \in \mathcal{S}$ .

Recalling that  $X_i$  is non-negative, it follows from (1) that the number of solutions of (5), with respect to  $m$ , is

$$\#\mathcal{T}_{x^*} = \frac{m^{k-1}}{(k-1)!} + \theta_2(k-1)m^{k-2}.$$

As  $m \leq n/d$ , we have  $(k-1)m^{k-2} \leq (k-1)(n/d)^{k-2}$ . Moreover, since  $x_i^* < d$ , we have also  $m = n/d + \theta_3 k$ . As  $k \geq 3$ , we obtain  $m = n/d + (3/2)\theta_4(k-1)$ . Since  $n/d \geq 1$ , Lemma 10 applies with  $c = 3/2$  giving:

$$\frac{m^{k-1}}{(k-1)!} = \frac{(n/d)^{k-1}}{(k-1)!} + \theta_5 \frac{e^{2/3}(3e/2)^{k-1}}{\sqrt{2\pi(k-1)}} (n/d)^{k-2}.$$

It follows that

$$(28) \quad \#\mathcal{T}_{x^*} = \frac{(n/d)^{k-1}}{(k-1)!} + \theta_6 \left( k-1 + \frac{e^{2/3}(3e/2)^{k-1}}{\sqrt{2\pi(k-1)}} \right) (n/d)^{k-2}.$$

Now, multiplying (27) and (28) together and using (23), we get finally

$$\begin{aligned} B_{k,d}(n) &= \Xi_{k,n}(d) \frac{(n/d)^{k-1}}{(k-1)!} + \Xi_{k,n}(d) \theta_7 \left( k-1 + \frac{e^{2/3}(3e/2)^{k-1}}{\sqrt{2\pi(k-1)}} \right) (n/d)^{k-2} \\ &= \Xi_{k,n}(d) \frac{(n/d)^{k-1}}{(k-1)!} + \theta_8 \left( k-1 + \frac{e^{2/3}(3e/2)^{k-1}}{\sqrt{2\pi(k-1)}} \right) n^{k-2} d. \end{aligned}$$

□

*Proof of Theorem 4.* Due to the cases discussed in the Introduction, we may assume that  $k \geq 3$ . Let  $n \geq e^{k2^{k+2}}$  and write

$$q(n) = \frac{\log n}{2} \quad \text{and} \quad d(n) = \prod_{p \leq q(n)} p.$$

From [14, Theorem 6], we have

$$\sum_{p \leq x} \log p < 1.001102x \quad \text{for all } x > 1.$$

Therefore

$$(29) \quad d(n) \leq \exp(1.001102q(n)) \leq n^{1.001102/2} < n^{0.5006}.$$

Let

$$\begin{aligned} \mathcal{B}'_k(n) &= \{(x_1, \dots, x_k) \in \mathcal{K}_{k-1} : \text{there exist } i, j \in \{1, \dots, k\} \text{ with } i \neq j, \text{ and } p \\ &\quad \text{with } p > q(n) \text{ and } p \mid \gcd(x_i, x_j)\} \end{aligned}$$

and  $B'_k(n) = \#\mathcal{B}'_k(n)$ . Using Definition 13, we have

$$\mathcal{B}_k(n) = \mathcal{B}_{k,d(n)}(n) \setminus \mathcal{B}'_k(n),$$

and, in particular,

$$(30) \quad B_k(n) = B_{k,d(n)}(n) + \theta_1 B'_k(n).$$

Estimating both  $B_{k,d(n)}(n)$  and  $B'_k(n)$ , we will see that the main part of  $B_k(n)$  is given by  $B_{k,d(n)}(n)$ .

First, we claim that

$$(31) \quad B'_k(n) = \theta_2 \frac{24n^{k-1}}{\log n}.$$

To get an element of  $\mathcal{B}'_k(n)$ , the pair  $\{i, j\}$  can be chosen in  $\binom{k}{2}$  ways. Once the pair  $\{i, j\}$  is chosen and the prime  $p > q(n)$  is fixed, we see that  $x_i$  and  $x_j$  are both multiples of  $p$  of magnitude at most  $n$ . Thus, the ordered pair  $(x_i, x_j)$  can be chosen in at most  $(n/p)^2$  ways. Once the pair  $(x_i, x_j)$  is chosen, we have

$$\sum_{1 \leq \ell \leq k, \ell \notin \{i, j\}} x_\ell = n - (x_i + x_j).$$

Therefore, the number of choices for the remaining summands  $x_\ell$  is the number of  $(k-2)$ -compositions of  $n - (x_i + x_j)$ , that is,

$$\binom{n - (x_i + x_j) - 1}{k-3} \leq \frac{n^{k-3}}{(k-3)!}.$$

Summing up, we obtain

$$B'_k(n) \leq \sum_{p>q(n)} \binom{k}{2} \frac{n^2}{p^2} \frac{n^{k-3}}{(k-3)!} \leq \frac{n^{k-1} k(k-1)}{2(k-3)!} \sum_{p>q(n)} \frac{1}{p^2}.$$

Observe now that

$$\begin{aligned} \sum_{p>q(n)} \frac{1}{p^2} &\leq \frac{1}{q(n)^2} + \int_{q(n)}^{\infty} \frac{dt}{t^2} = \frac{1}{q(n)^2} + \left(-\frac{1}{t}\right) \Big|_{q(n)}^{\infty} \\ (32) \quad &= \frac{1}{q(n)^2} + \frac{1}{q(n)} \leq \frac{4}{\log n}. \end{aligned}$$

This gives

$$B'_k(n) \leq \frac{4k(k-1)}{2(k-3)!} \frac{n^{k-1}}{\log n} \leq \frac{24n^{k-1}}{\log n},$$

where for the last inequality we used the fact that

$$\frac{2k(k-1)}{(k-3)!} \leq 24 \quad \text{for all } k \geq 3,$$

which proves (31).

Next, we estimate  $B_{k,d(n)}(n)$ . Using Proposition 15, we have

$$(33) \quad B_{k,d(n)}(n) = \Xi_{k,n}(d(n)) \frac{(n/d(n))^{k-1}}{(k-1)!} + \theta_3 \left( k-1 + \frac{e^{2/3}(3e/2)^{k-1}}{\sqrt{2\pi(k-1)}} \right) n^{k-2} d(n).$$

Extending the product from the main term  $M$  of (33) to all primes, we get

$$\begin{aligned} M &= \frac{n^{k-1}}{(k-1)!} \frac{\Xi_{k,n}(d(n))}{d(n)^{k-1}} = \frac{n^{k-1}}{(k-1)!} \prod_{p|n, p \leq q(n)} \frac{\eta_k(p)}{p^{k-1}} \prod_{p \nmid n, p \leq q(n)} \frac{\delta_k(p)}{p^{k-1}} \\ &= \frac{n^{k-1}}{(k-1)!} \prod_{p|n} \frac{\eta_k(p)}{p^{k-1}} \prod_{p|n, p > q(n)} \frac{p^{k-1}}{\eta_k(p)} \prod_p \frac{\delta_k(p)}{p^{k-1}} \prod_{p|n} \frac{p^{k-1}}{\delta_k(p)} \prod_{p \nmid n, p > q(n)} \frac{p^{k-1}}{\delta_k(p)} \\ &= D_k g_k(n) \frac{n^{k-1}}{(k-1)!} \prod_{p|n, p > q(n)} \frac{p^{k-1}}{\eta_k(p)} \prod_{p \nmid n, p > q(n)} \frac{p^{k-1}}{\delta_k(p)} \\ (34) \quad &= D_k g_k(n) \frac{n^{k-1}}{(k-1)!} E_k(n), \end{aligned}$$

(where in the third equality we used the relation (3) between  $\eta_k$  and  $\delta_k$ ). We now estimate the error term

$$E_k(n) = \prod_{p|n, p > q(n)} \frac{p^{k-1}}{\eta_k(p)} \prod_{p \nmid n, p > q(n)} \frac{p^{k-1}}{\delta_k(p)}.$$

Since  $q(n) \geq k2^{k+1} > \sqrt{2^k k}$ , from Lemma 8 f) and g), we get that

$$0 \leq \max \left\{ \log \left( \frac{p^{k-1}}{\eta_k(p)} \right), \log \left( \frac{p^{k-1}}{\delta_k(p)} \right) \right\} \leq \log \left( 1 + \frac{k2^k}{p^2} \right) \leq \frac{k2^k}{p^2},$$

for any  $p > q(n)$ . In particular,  $E_k(n) \geq 1$  and using (32), we get

$$0 \leq \log(E_k(n)) \leq k2^k \sum_{p>q(n)} \frac{1}{p^2} \leq \frac{2^{k+2}k}{\log n} \leq 1.$$

Recalling that  $e^y \leq 1 + 2y$  for any  $0 \leq y \leq 1$ , we reach finally

$$(35) \quad E_k(n) = 1 + \theta_4 \frac{2^{k+3}k}{\log n}.$$

Now we go back to the main term  $M$ . By Theorem 5,  $1/(2k) < g_k(n) < 2k$  and  $D_k < 1$ . Then, using (34) and (35), we obtain

$$(36) \quad \begin{aligned} M &= D_k g_k(n) \frac{n^{k-1}}{(k-1)!} \left( 1 + \theta_4 \frac{2^{k+3}k}{\log n} \right) \\ &= D_k g_k(n) \frac{n^{k-1}}{(k-1)!} + \theta_5 \frac{2^{k+4}k^2}{(k-1)! \log n} n^{k-1} \\ &= D_k g_k(n) \frac{n^{k-1}}{(k-1)!} + 682.7 \theta_6 \frac{n^{k-1}}{\log n}, \end{aligned}$$

where we used the fact that

$$\frac{2^{k+4}k^2}{(k-1)!} \leq 682.7 \quad \text{for all } k \geq 3.$$

We now estimate the error in (33). First of all observe that since  $\log n \leq n^\alpha$  holds for any  $\alpha \geq e^{-1}$ , we surely have  $\log n \leq n^{0.3994}$  and thus, using (29),

$$n^{k-2}d(n) \leq n^{k-1}n^{-0.4994} = \frac{1}{n^{0.1}} \frac{\log n}{n^{0.3994}} \frac{n^{k-1}}{\log n} \leq \frac{1}{n^{0.1}} \frac{n^{k-1}}{\log n}.$$

Since  $n \geq e^{k2^{k+2}}$ , we have

$$\left( k - 1 + \frac{e^{2/3}(3e/2)^{k-1}}{\sqrt{2\pi(k-1)}} \right) \frac{1}{n^{0.1}} \leq \frac{k - 1 + \frac{e^{2/3}(3e/2)^{k-1}}{\sqrt{2\pi(k-1)}}}{e^{\frac{k2^{k+2}}{10}}},$$

which is a decreasing function of  $k$  whose values is always strictly less than 0.002.

This says that the error in (33) can be written as  $0.002\theta_7 n^{k-1}/\log n$ . Summing up, considering (30), (31) and (36) and noting that  $682.7 + 24 + 0.002 < 707$ , we find

$$B_k(n) = D_k g_k(n) \frac{n^{k-1}}{(k-1)!} + 707\theta_8 \frac{n^{k-1}}{\log n},$$

which is what we wanted.  $\square$

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